



# Playing with polytopes

## NOTABLE MATHS PROBLEMS

### COLLATZ CONJECTURE

Take any positive integer; if even, divide by two; if odd, triple it and add one. Using each result as the input for the next step, no matter what number starts the sequence, the end result is 1.

**Simply:** That all paths in a certain kind of number sequence eventually lead to 1.

**Also known as:** The  $3n + 1$  conjecture, the Ulam conjecture (after Polish mathematician Stanislaw Ulam), the Syracuse problem or HOTPO (Half Or Triple Plus One) in computer programming.

**Discipline:** Number theory.

**Originator:** German mathematician Lothar Collatz, 1937; he made little progress and published a history of its origin much later.

**Incentive:** \$US50 by H Coxeter, 1970; then \$US500 by Hungarian mathematician Paul Erdős; £1,000 by B Thwaites, 1996; solving a problem with a tantalizingly elementary form that has eluded top mathematicians.

**Examples:** If  $n = 6$ , the number sequence is 6, 3, 10, 5, 16, 8, 4, 2, 1. If  $n = 27$ , the sequence takes 111 steps, climbing to over 9,000 before descending to 1.

**Explorations:** Many attempts have been made to settle the conjecture using technologies from number theory, dynamical systems and Markov chains. USA mathematician Jeffrey Lagarias proved in 1985 that the problem has no nontrivial cycles of length  $< 275,000$ . Another approach took the opposite direction; instead of proving that all natural numbers eventually lead to 1, it proved that 1 leads to all natural numbers. USA mathematician John Conway proved in 1972 that Collatz-type problems can be formally undecidable.

**State of play:** The conjecture has not been proved, but most mathematicians who have worked on it believe it to be true. The conjecture has been checked by computer for all starting values up to  $10 \times 2^{58}$ . However, some important conjectures have been found to be false only with very large counterexamples.

Each odd number in Collatz sequences is on average  $\frac{3}{4}$  of the previous one, leading to an argument that every Collatz sequence should decrease in the long run. This is also not a proof because it pretends that sequences are assembled from uncorrelated probabilistic events.

Like many mathematicians working with symmetries, Dr Isabel Hubard started working with polyhedra because she loved their beautiful shapes. "I like to draw pictures, use different colours and study them for hours. Playing like a five-year-old, I can see which mathematical properties they satisfy." Hubard prints out her own triangular and hexagonal graph paper for doodling.

Hubard, who hails from Mexico City, touched down in Aotearoa for 2008, after a Masters and PhD in York University in Toronto, and before a post-doctoral position in Brussels, Belgium. She is a temporary lecturer at the University of Auckland in graph theory.

Mathematics orders the relationships of vertices to edges and faces in polyhedra. "A vertex is smaller than an edge if it is one of the endpoints; an edge is smaller than a face if it belongs to the face. Then we can forget about what they look like physically and think of them as objects with an order between them. We can use as many layers or ranks as we like. We say the faces are of rank 2 to any finite number; they are abstract polytopes, unable to be built in the 3-space we occupy."

Regular polyhedra - such as tetrahedra, cubes and octahedra - are the most studied; they include many symmetrical reflections and rotations. Other polytopes that have all the possible rotations, but not reflections, are called chiral. A related vertex, edge and face on all polytopes is called a flag.

"When polytopes are regular, the number of symmetries they have is the same as the number of flags," says Hubard. "And the reverse is true for finite polytopes - if the number of flags is the same as the symmetries, then the polytope is regular."

"For finite chiral polytopes, the number of symmetries is half the number of flags. But the reverse doesn't work - if you have half the symmetries, it doesn't mean that it's chiral." Hubard studied two-orbit polytopes for her PhD.

Symmetries can divide the flags into two sets, called orbits. "Given a two-orbit polyhedron, we can tell the group of symmetries, with its generators and some relations they will satisfy. If you give me any symmetrical group, with generators and relations, I can tell you if it will be a group of two-orbit polyhedra and if it is, I can construct it."

She also worked on what happens to polyhedra when the order of vertices, edges and faces for each shape is reversed. "When we reverse the order, we get a different polytope. A cube becomes an octahedron and a dodecahedron becomes an icosahedron. But a tetrahedron becomes a tetrahedron in a different place; this is called self-dual."

"It is intuitive to think that if we reverse the order twice, we get the same object. The problem is that the same object may end up in a different position." After two reversals, regular polytopes can always return to the same position; those with very little symmetry don't always return to the same position. Hubard and her supervisor Asia Weiss found that chiral polytopes of odd ranks can always come back to their initial position, but that some of those of rank 4 cannot. Later, with Alen Orbanic, Hubard and Weiss found a rule that decided for any polytope.

She is now working on generalising from two-orbit polytopes to those of any rank, with Professor Egon Schulte at North Eastern University in Boston. And when she's not doing maths, she's teaching tango in downtown Auckland.