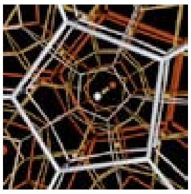
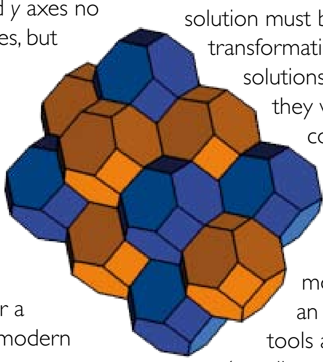
Tiling in curved space



Professor Gaven Martin is in full flight, describing how non-linear partial differential equations explain how the microstructure in a steel bar affects the macrosopic distribution of force as it is bent. If his listener doesn't understand, he takes a step back and sketches the graph of a simple function on his office whiteboard at Massey University in Albany.

He shows how finding its minima leads to a differential equation. The *x* and *y* axes no longer represent single variables, but states – which are functions in themselves. So the energy of the bar is now a function of functions. The solution of the partial differential equation (PDE) will be the particular bend that, out of all possible twists, uses the least energy.



This PDE might be solvable for a simple steel bar, but for most modern materials and structures, such as cars, bridges, tower blocks and even bars made from composite materials, other PDEs are not. "So we build models of the key energy functionals of those structures - their crystalline alignment or the interface between two different components - and then minimise them, leading often to quite complicated non-linear PDEs."

"Pure mathematicians study classes of differential equations with generic features to validate these models. Generic features include whether the equation has a solution or not," he says. "This sounds almost useless, but it's not. It's a damn sight better searching for something if you know it's there. Other important features include whether there are multiple solutions or only one and whether the solution is regular or not, because otherwise a computer has no chance of finding it. The solutions to differential equations can be very irregular."

In 1850 Joseph Liouville hypothesised that the only conformal transformations of space (the symmetries of physical theories such as relativity) are the usual Lorentz or Mōbius transformations. Liouville proved this if the symmetry was sufficiently smooth. "But this begs the question," says Martin. "Physical systems exist where solutions are highly irregular. Liouville was effectively asking if there were any physical symmetries other than the obvious ones." In 1990, with Polish mathematician Tadeusz Iwaniec, Martin completely solved this problem.

"We showed *exactly* how regular a solution must be so that it was a Mōbius transformation; we also constructed solutions, functions, that weren't. But they were so irregular that they couldn't represent a physical symmetry – they blew up all over the place." This result remains the only fully non-linear system of PDEs in more than two dimensions with an exact regularity theory. The tools and ideas created in the proof (non-linear Hodge theory) opened a theoretical bottleneck and led to hundreds of mathematical papers being published.

Martin's description of a new result in hyperbolic (curved space) geometry leads to another flurry of whiteboard drawings. This time they are spheres, tori (donut shapes) and other surfaces with more holes and the two-dimensional polygons that tessellate their surfaces.

Tessellations are periodic tilings of two or more dimensions that cover a plane, like graph paper, or fill a multi-dimensional shape, like honeycomb, with no overlaps and no gaps. The result Martin is describing is the formula for the smallest possible three-dimensional tessellation of hyperbolic space – solving a problem first posed by Carl Siegel in 1945 after he had solved the 2D problem.

Every surface can be identified with a hyperbolic tessellation and recently results of Grisha Perelman (of Poincaré conjecture fame) show the same is more or less true in three dimensions. Martin, with Fred Gehring and former PhD student Tim Marshall, showed that tetrahedrons with internal angles ∏/2, ∏/3 and ∏/5 gave the smallest possible 3-D hyperbolic tessellation and found deep connections with number theory.

"Sometimes these simple problems take decades." Martin worked on the 3-D tessellation problem for nearly 20 years, getting partial results in small steps. "You get the occasional blinding flashes of insight, but they often come when you're younger."

Martin met his future wife Dianne, a biology professor at Massey University, when they attended Henderson Intermediate. He did his PhD at the University of Michigan and has worked in the USA, Finland, Sweden, France and Australia and at the University of Auckland. Martin was one of the NZIMA's original principal investigators, and represents the New Zealand Mathematics Research Institute (Inc.) on the NZIMA board.

Above: Gaven Martin; photo courtesy of Massey University. **Centre: A bi-truncated cubic honeycomb.**